A Practical Formulation for an Anisotropic and Nonstationary Matérn Class Correlation Operator

A Brief History of Correlation Operators in Variational Data Assimilation

Timothy Smith
Patrick Heimbach

ECCO Meeting
Jan. 25, 2023

Data Assimilation Context

$$\mathcal{J}(\delta\mathbf{v}) = \frac{1}{2}\underbrace{\left\|\mathbf{R}^{-1/2}\left(f(\delta\mathbf{v}) - \mathbf{d}\right)\right\|_2^2}_{\text{misfit}} + \frac{1}{2}\underbrace{\|\delta\mathbf{v}\|_2^2}_{\text{prior}}$$

$$\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$$

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$$\mathbf{B}^{1/2} := \mathbf{\Sigma}\mathbf{C}$$

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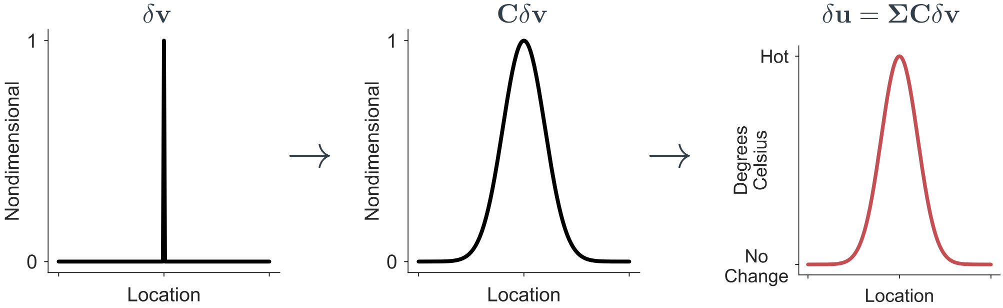
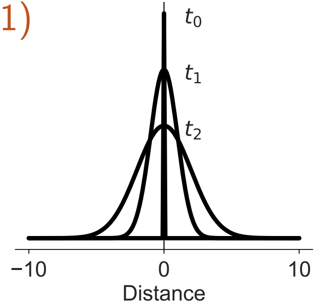


Explicit Diffusion: Weaver and Courtier (2001)

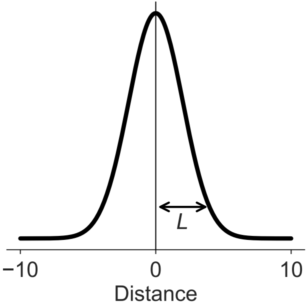
$$\frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \mathbf{K} \nabla \mathbf{v}$$

$$\mathbf{v}(T) \simeq (I + \nabla \cdot \mathbf{K} \nabla)^N \, \mathbf{v}(0)$$

Good:



Bad:



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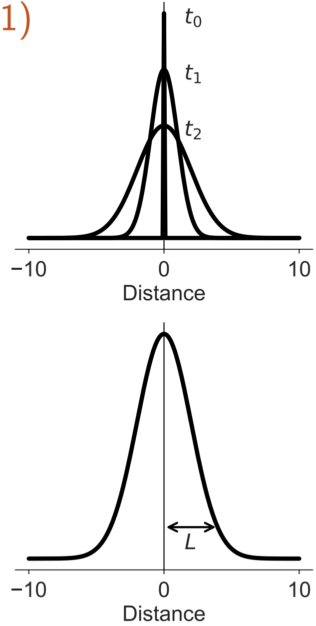

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- Easy to implement, diffusion already in GCMs
- Handles boundaries naturally
- Imposes Gaussian correlation structure
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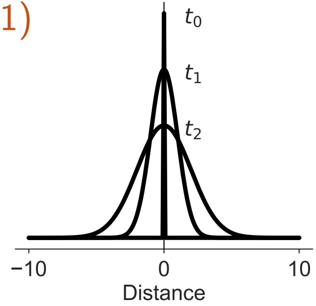
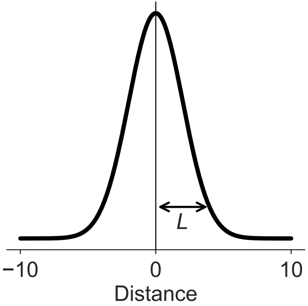
- Necessary but insufficient conditions for N, usually N has to be large





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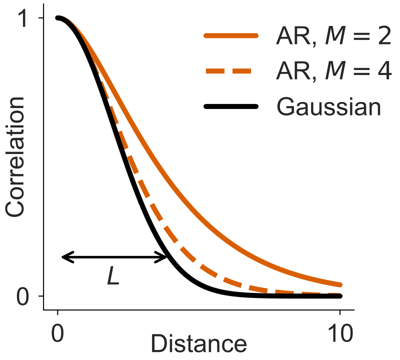
- Access to correlation and inverse, see Guillet et al. (2019)



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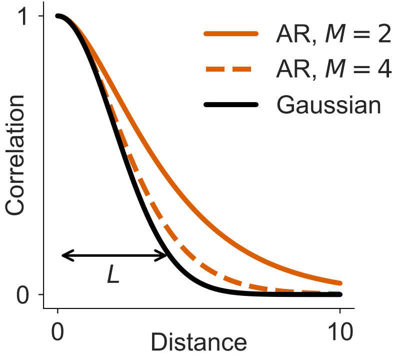

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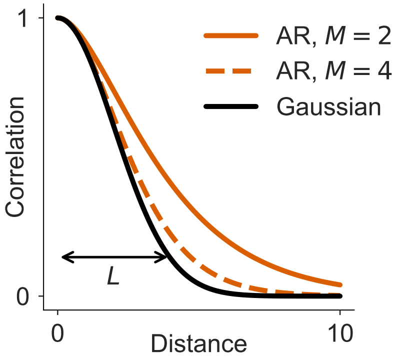
- Even with fixed L, inconsistent correlation length scales with different M



Matérn SPDE: Lindgren et al. (2011)

$$\mathbf{v} = (\delta - \nabla \cdot \nabla)^{-M} \mathbf{z}$$

$$\mathbf{z} \sim \mathcal{N}(0, I)$$
$$\mathbf{v} \sim \mathcal{N}(0, \mathbf{C}\mathbf{C}^T)$$

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- Consistently achieve correlation ~ 0.14 at specified length scale
- Length scale specified in $\delta = \delta(L)$

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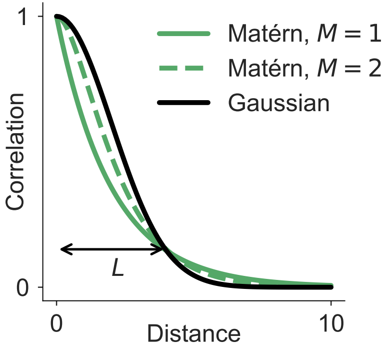

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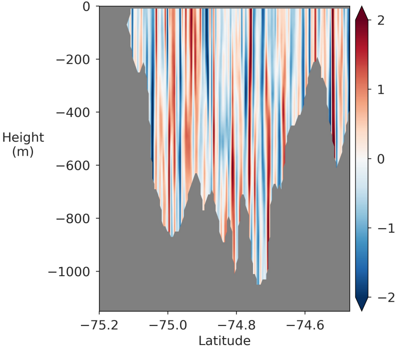

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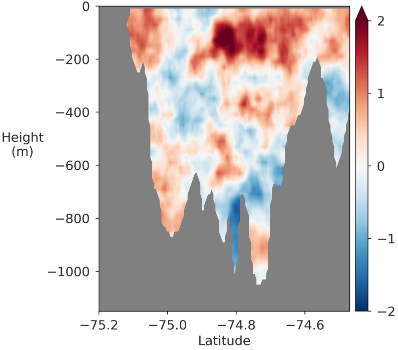

Mapping Method

Define mapping $\varphi(\mathbf{x})$, through its Jacobian, $\Phi(\mathbf{x})$
Length scale specified in $\delta(L)$ and $\Phi(\mathbf{x}; L)$

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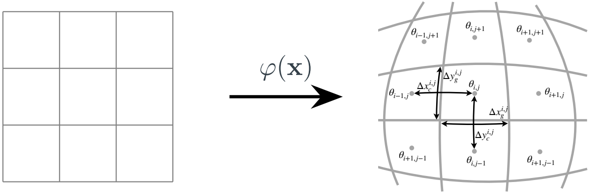

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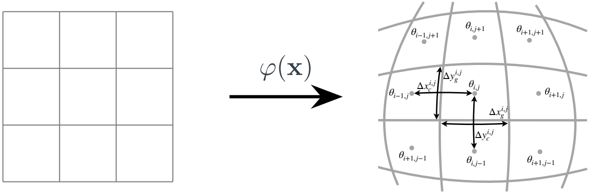

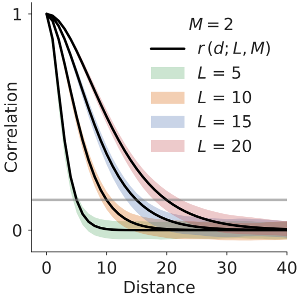

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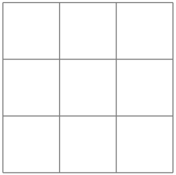
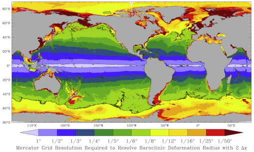
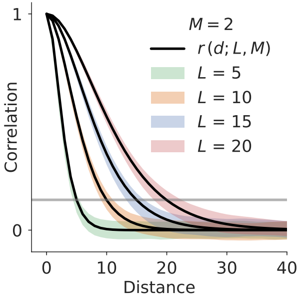


$\varphi(\mathbf{x})$





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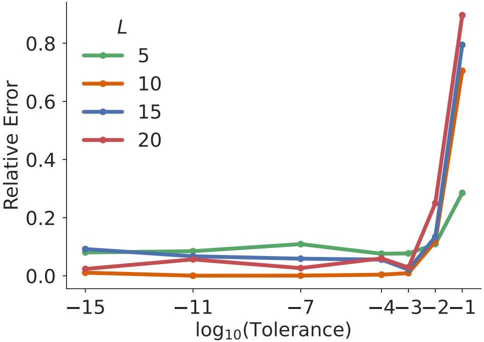
Precision & Speed in Global LLC90 Domain



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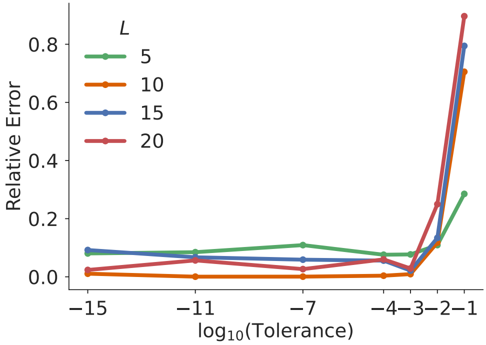

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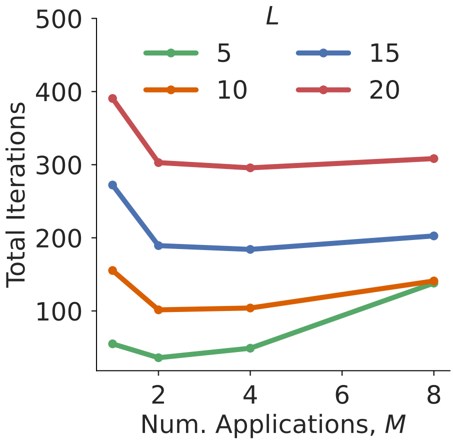


Increasing number of operator applications, M, *improves* convergence rate!

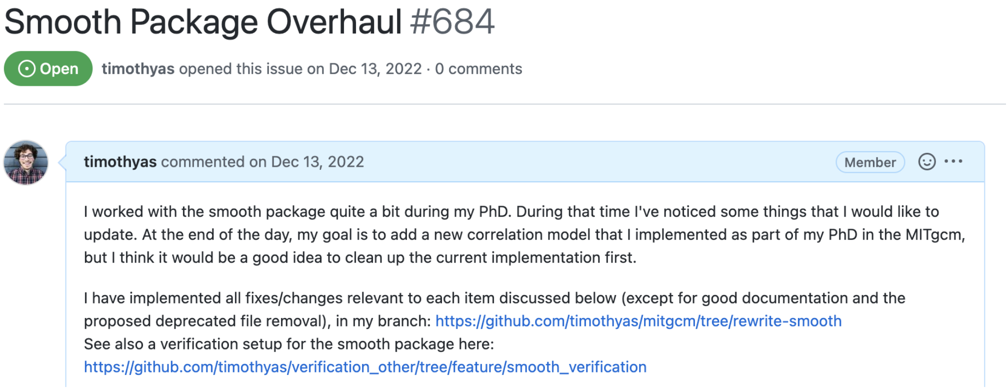
Summary

	Reference	Boundaries & GCM Friendly	Fixed Iterations (N)	Access to Inverse	Flexible Shape (M)	Consistent Correlation Length
Explicit Diffusion	Used by ECCO (Weaver & Courtier, 2001)	✅	❌	❌	🤷‍♂️	✅
Implicit Diffusion	(Mirouze & Weaver, 2010)	✅	✅	✅	✅	❌
Mapped Matérn	(Preprint) (Smith, 2022)	✅	✅	✅	✅	✅

Coming Soon to the MITgcm...

Comments are appreciated!

Either on GitHub or via email at `tim.smith@noaa.gov`



Smooth Package Overhaul #684

Open timothyas opened this issue on Dec 13, 2022 · 0 comments

timothyas commented on Dec 13, 2022 Member

I worked with the smooth package quite a bit during my PhD. During that time I've noticed some things that I would like to update. At the end of the day, my goal is to add a new correlation model that I implemented as part of my PhD in the MITgcm, but I think it would be a good idea to clean up the current implementation first.

I have implemented all fixes/changes relevant to each item discussed below (except for good documentation and the proposed deprecated file removal), in my branch: https://github.com/timothyas/mitgcm/tree/rewrite-smooth
See also a verification setup for the smooth package here:
https://github.com/timothyas/verification_other/tree/feature/smooth_verification

Figure: `github.com/MITgcm/MITgcm/issues/684`

References I

Guillet, O., Weaver, A. T., Vasseur, X., Michel, Y., Gratton, S., and Gürol, S. (2019). Modelling spatially correlated observation errors in variational data assimilation using a diffusion operator on an unstructured mesh. *Quarterly Journal of the Royal Meteorological Society*, 145(722):1947–1967. _eprint: https://rmets.onlinelibrary.wiley.com/doi/pdf/10.1002/qj.3537.

Lindgren, F., Rue, H., and Lindström, J. (2011). An explicit link between gaussian fields and gaussian markov random fields: the stochastic partial differential equation approach. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 73(4):423–498.

Mirouze, I. and Weaver, A. T. (2010). Representation of correlation functions in variational assimilation using an implicit diffusion operator. *Quarterly Journal of the Royal Meteorological Society*, 136(651):1421–1443. _eprint: https://rmets.onlinelibrary.wiley.com/doi/pdf/10.1002/qj.643.

Smith, T. A. (2022). A Practical Formulation for an Anisotropic and Nonstationary Matérn Class Correlation Operator. preprint, Oceanography.

Weaver, A. T. and Courtier, P. (2001). Correlation modelling on the sphere using a generalized diffusion equation. *Quarterly Journal of the Royal Meteorological Society*, 127(575):1815–1846.