ECCO summer school 2019

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Ocean Modeling, part I

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Plan

Part I

- Ocean Model equations
- Discretised equations, mainly focus on MITgcm formulation
- some modeling recipe (stability, accuracy, conservation)

Part II

- Software architecture
- Forcing and diagnostics
- interface with SGS parameterisation and other components

Continuous set of equations

Hydrostatic, boussinesq, primitive equation in height-coordinate:

1) Simplified Equation of State (EOS) $\rightarrow$ **incompressible**:

$$\text{density :} \quad \rho = \rho' + \rho_c \simeq \rho(\theta, S, p_o(z))$$

2) Use constant ρ_c in place of ρ everywhere except in gravity term $\rightarrow$ **boussinesq**

3) Reduce vertical momentum equation to hydrostatic balance ($\epsilon_{nh} = 0$) $\rightarrow$ **hydrostatic**

$$\nabla_h \cdot \mathbf{v}_h + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

$$\frac{D\mathbf{v}_h}{Dt} + f\hat{\mathbf{k}} \times \mathbf{v}_h + \frac{1}{\rho_c}\nabla_h p = \frac{1}{\rho_c}\mathcal{F}_{\mathbf{v}} \qquad (2)$$

$$g\rho' + \frac{\partial p'}{\partial z} = 0 + \epsilon_{nh}\left(\mathcal{F}_w - \rho_c\frac{Dw}{Dt}\right) \qquad (3)$$

$$\rho' = \rho(\theta, S, p_o(z)) - \rho_c \qquad (4)$$

$$\frac{D\theta}{Dt} = \frac{1}{\rho_c C_p}\mathcal{H}_\theta \qquad (5)$$

$$\frac{DS}{Dt} = \frac{1}{\rho_c}\mathcal{Q}_s \qquad (6)$$

where $\frac{D}{Dt}() = \frac{\partial}{\partial t}() + \mathbf{v}\cdot\nabla()$; $\mathbf{v} = (u, v, w) = (\mathbf{v}_h, w)$;

Free surface

Boundary conditions at surface ($z = \eta$) and bottom ($z = -H$):

$$w_{(z=\eta)} = \frac{D\eta}{Dt} - \frac{1}{\rho_c}(P - E) \quad ; \quad w_{(z=-H)} = -\mathbf{v}_h \cdot \nabla H$$

combine with (1):

$$\frac{\partial \eta}{\partial t} + \nabla \int_{-H}^{\eta} \mathbf{v}_h dz = \frac{1}{\rho_c}(P - E) \tag{7}$$

$$\text{and in (2)}: \quad \nabla_h p = \nabla_h \left(g\rho_c \eta - g\rho_c z - \int g\rho' dz \right) = \rho_c g \nabla \eta + \nabla_h p'$$

Solving numerically

- Discretise in space

  choice of horizontal and vertical grid $\Leftrightarrow$ increment in space $\Delta x, \Delta y, \Delta z$
  for each variable ϕ, one value at each grid cell $\phi_{i,j,k}$

- Discretise in time
  choice of a time increment Δt
  evolution of variable ϕ represented as ϕ^n at time $t = n\Delta t$

Ideally: chose resolution in space and time according to processes of interest

Practically: spacial resolution is limited by computer resources while
Δt is generally limited by stability criteria.

$\Rightarrow$ parameterisation to account for unresolved Sub-Grid Scale (SGS) processes

Time stepping schemes

- Forward Euler time-stepping ($1^{rst}O$):
  $(\phi^{n+1} - \phi^n)/\Delta t = \frac{\partial \phi}{\partial t}\Big|^n$

- Adams-Bashforth, second order (AB-2):
  $(\phi^{n+1} - \phi^n)/\Delta t = \frac{3}{2}\ \frac{\partial \phi}{\partial t}\Big|^n - \frac{1}{2}\ \frac{\partial \phi}{\partial t}\Big|^{n-1}$

- Backward Euler time-stepping ($1^{rst}O$):
  $(\phi^{n+1} - \phi^n)/\Delta t = \frac{\partial \phi}{\partial t}\Big|^{n+1}$
  generally, $\frac{\partial \phi}{\partial t} = fct(\phi) \rightarrow$ implicit method

- Crank-Nicolson time-stepping ($2^{nd}O$, implicit method):
  $(\phi^{n+1} - \phi^n)/\Delta t = \frac{1}{2}\ \frac{\partial \phi}{\partial t}\Big|^n + \frac{1}{2}\ \frac{\partial \phi}{\partial t}\Big|^{n+1}$

Wide range of oceanic time-scales, use different scheme for each term (depending on stability, precision and complexity).

Influence:

- how the code is organized
- how each term is computed ($\rightarrow$ diagnostics)

Simple illustration: 2-D advection of passive tracer

Tracer T advected by non-divergent 2-D flow: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} = -\frac{\partial u.T}{\partial x} - \frac{\partial v.T}{\partial y} \qquad \text{advective form / flux form}$$

- discretise in space $(\Delta x, \Delta y)$
  the continuity equation: $\delta^i(u\Delta y) + \delta^j(v\Delta x) = 0$
  and using centered $2^{nd}O$ advection scheme:

$$G_{i,j} = \left.\frac{\partial T}{\partial t}\right|_{(i,j)} = -\frac{1}{\Delta x \Delta y}\left(\delta^i(u\Delta y\,\overline{T}^i) + \delta^j(v\Delta x\,\overline{T}^j)\right)$$

- discretise in time (Δt) using quasi AB-2 (e.g., $\alpha_{AB} = 0.55$):

$$T^{n+1}_{i,j} - T^n_{i,j} = \Delta t\,\big(\,(1+\alpha_{AB})G^n_{i,j} - \alpha_{AB}\,G^{n-1}_{i,j}\,\big)$$

Time stepping choice

- External mode: $\partial\eta/\partial t$ and $-g\nabla\eta$
  fast mode: use unconditionally stable scheme (implicit):
  - backward Euler (damp fast, un-resolved adjustment)
  - Crank-Nicolson (energy conserving)
- Momentum advection $G_{\mathbf{v}h}^{adv} = -\mathbf{v}\cdot\nabla\mathbf{v}_h$
  and Coriolis term:
  for precision (energy conservation) and stability, use AB-2 (or AB-3)
- Viscous/Dissipation term $G_{\mathbf{v}h}^{visc} = -\nabla\cdot(-\nu\nabla\mathbf{v}_h)$
  use AB-2 (precision), Euler forward (more stable), or/and Backward (implicit) for the vertical part.
- Internal modes: Tracer advection and $-1/\rho_c\nabla p'$
  - AB-2 and synchronized time-stepping
  - Direct Space and Time (DST) tracer advection scheme with staggered time-stepping (more stable)

Surface pressure implicit method

backward time-stepping for surface pressure gradient in (2):

$$\mathbf{v}_h^{n+1} = \mathbf{v}_h^{*} - \Delta t\, g \nabla \eta^{n+1} \tag{8}$$

with

$$\mathbf{v}_h^{*} = \mathbf{v}_h^{n} - \frac{\Delta t}{\rho_c} \nabla p'^{(n+1/2)} + \Delta t \left[\left(G_{\mathbf{v}}^{adv,cori} \right)^{AB} + G_{\mathbf{v}}^{visc}(n) + \frac{1}{\rho_c} \mathcal{F}_{\mathbf{v}}^{n} \right]$$

and backward time-stepping of transport in (7):

$$\eta^{n+1} = \eta^{n} - \Delta t \nabla \cdot \int_{-H}^{\eta^n} \mathbf{v}_h^{n+1} dz + \frac{\Delta t}{\rho_c} (P - E)$$

Using (8) to replace $\mathbf{v}_h^{n+1}$ above:

$$\eta^{n+1} - g(\Delta t)^2 \nabla \cdot (H + \eta^n) \nabla \eta^{n+1} = \eta^{n} - \Delta t\, \nabla \cdot \int_{-H}^{\eta^n} \mathbf{v}_h^{*} dz + \frac{\Delta t}{\rho_c} (P - E)$$

Solve iteratively using conjugate gradient method (cg2d)

$\rightarrow$ get η^{n+1} ; replace in (8) to get $\mathbf{v}_h^{n+1}$

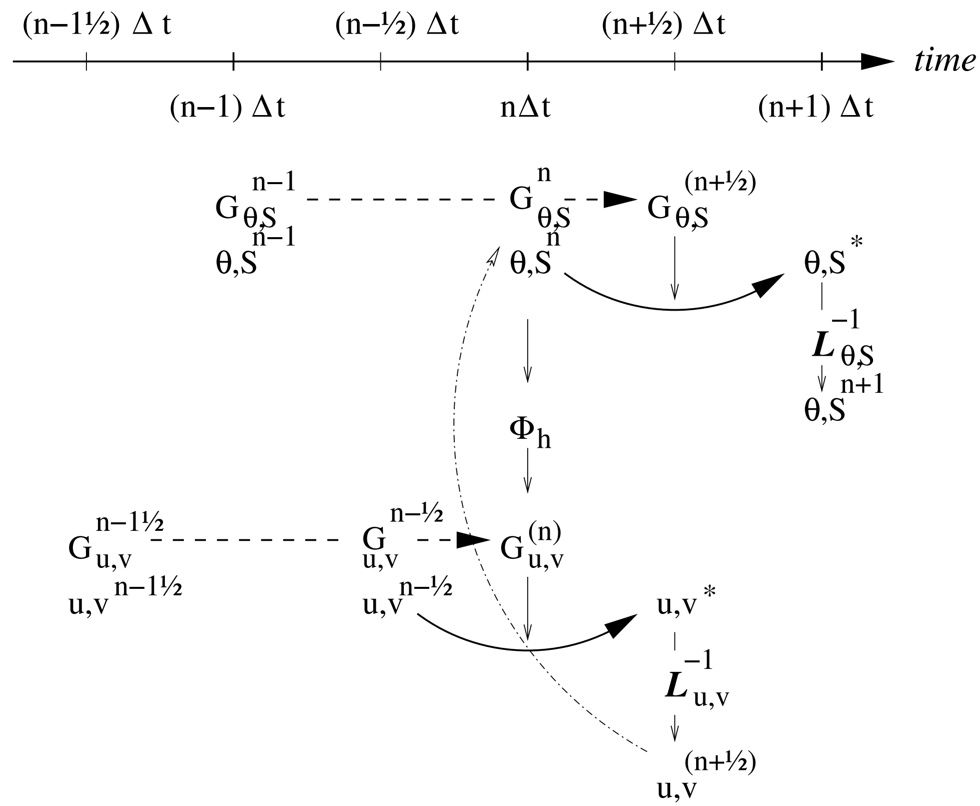
Staggered time-stepping

Used in ECCO set-ups:

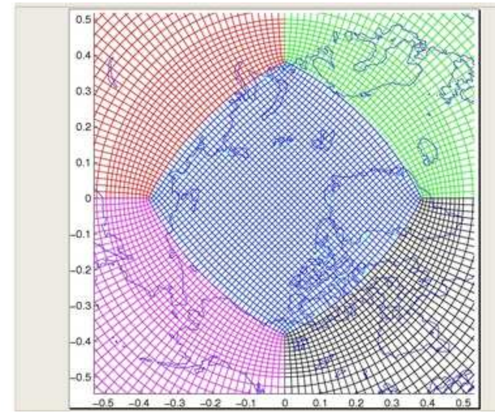
1) $\mathbf{v}_h^{n-1/2} \rightarrow \mathbf{v}_h^{n+1/2}$ using $\mathrm{AB}[G_v]^{(n)}$
and p' from θ^n, S^n

2) $(\theta^n, S^n) \rightarrow (\theta^{n+1}, S^{n+1})$ using
(2a) $\mathrm{AB}[G_{(\theta,S)}]^{(n+1/2)}$
(2b) DST advection scheme
$\mathrm{Adv}\left(\mathbf{v}^{n+1/2}, (\theta^n, S^n), \Delta t\right)$

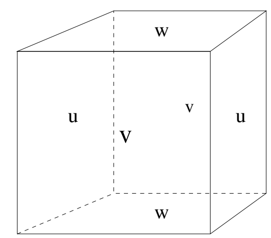
3) Backward time stepping on Linear term,
e.g., vertical viscosity and diffusion:
Invert 3 diagonal operator $\mathbf{L}_{\mathbf{v}h}, \mathbf{L}_{\theta,S} \rightarrow$
$\mathbf{L}_{\mathbf{v}h}^{-1}, \mathbf{L}_{\theta,S}^{-1}$



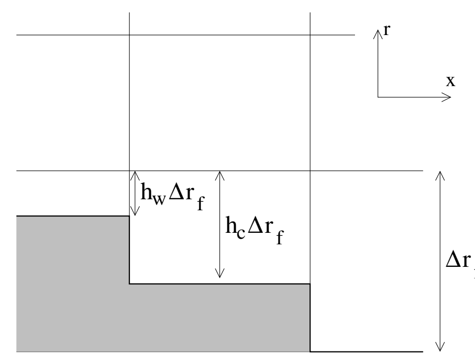
Discretisation in space



- curvilinear horizontal grid, locally orthogonal



- staggered variables on Arakawa C grid
  θ, S, p' at grid-cell center ; u, v, w at grid-cell faces



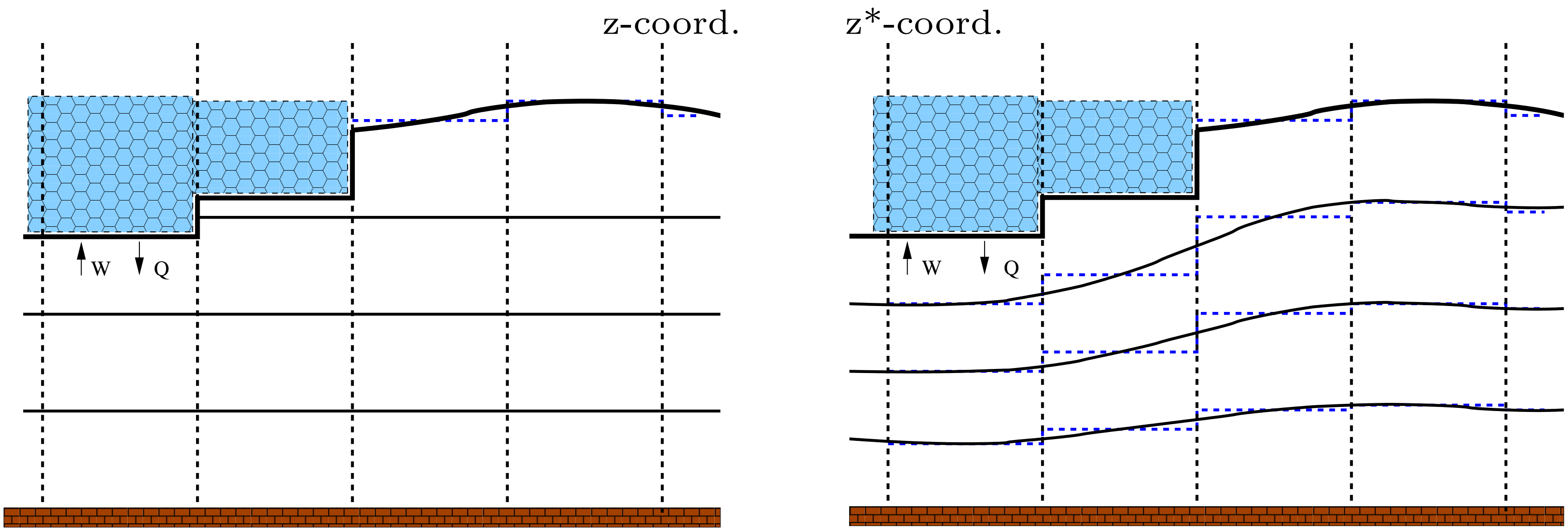
- bathymetry with partial cell

- finite volume method:
  budget integrated over a grid-cell

re-scaled vertical coordinate z^*

$$z = \eta + z^* \frac{H + \eta}{H}$$

- vertical coordinate follows free-surface displacement
- stretch/squeeze level thickness (ratio: $(H + \eta)/H$)
- in z^* coordinate, model domain is fixed, from $z^* = -H$ to $z^* = 0$



Volume and Tracer equation

Grid-cell face area: A_x, A_y, A_z (e.g., $A_x = h_W^{fac} \Delta r_F \Delta x_G$),
grid cell volume: $\mathcal{V} = A_z \Delta z$ (with $\Delta z = h_C^{fac} \Delta r_F$),
volume transport: $U = A_x u$; $V = A_y v$; $W = A_z w$

$$\begin{aligned}
\Delta z^{n+1} - \Delta z^n &= -\frac{\Delta t}{A_z}(\delta^i U + \delta^j V - \delta^k W) \\
(\Delta z S)^{n+1} - (\Delta z S)^n &= -\frac{\Delta t}{A_z}\left(\delta^i(\widehat{U.S^n}^i) + \delta^j(\widehat{V.S^n}^j) - \delta^k(\widehat{W.S^n}^k)\right)
\end{aligned}$$

Tracer (here S) transport fluxes: $\widehat{U.S^n}^i, \widehat{V.S^n}^j, \widehat{W.S^n}^k$ function of selected advection scheme, e.g., with $2nd$ order centered:
$\widehat{U.S^n}^i = U \cdot \overline{S^n}^i = U(S^n_{i-1} + S^n_i)/2$

Note: in z-coordinate, $\Delta z^{n+1} = \Delta z^n$ everywhere except at the surface (non-linear free-surface); with z^*, Δz varies everywhere according to $\frac{\partial \eta}{\partial t}$:

$$\Delta z^{n+1} - \Delta z^n = (\eta^{n+1} - \eta^n)\frac{\Delta z^*}{H}$$

Momentum equation

- Flux form

$$\frac{\partial \mathbf{v}_h}{\partial t} + [\nabla \cdot \mathbf{v}]\, \mathbf{v}_h + f\hat{\mathbf{k}} \times \mathbf{v}_h = -g\nabla\eta - \frac{1}{\rho_c}\nabla_h p' + \nabla \cdot (\nu \nabla \mathbf{v}_h) + \frac{1}{\rho_c}\mathcal{F}_{\mathbf{v}}$$

for curvature of horizontal grid, requires to compute and add metric terms

- Vector invariant form (no metric term)

$$\frac{\partial \mathbf{v}_h}{\partial t} + (f + \zeta)\hat{\mathbf{k}} \times \mathbf{v}_h + \nabla \text{KE} + w\frac{\partial}{\partial z}\mathbf{v}_h = -g\nabla\eta - \frac{1}{\rho_c}\nabla_h p' + \nabla \cdot (\nu \nabla \mathbf{v}_h) + \frac{1}{\rho_c}\mathcal{F}_{\mathbf{v}}$$

with vorticity: $\zeta = \nabla \times \mathbf{v}_h$ and kinetic energy: $\text{KE} = (u^2 + v^2)/2$

see MITgcm manual (https://mitgcm.readthedocs.io/en/latest/) for detailed discretisation in space of these 2 momentum formulations

Stability Criteria

Based on linear analysis:

- CourantFriedrichsLewy (CFL) number, per process:
  advection: $\mathrm{CFL}^{adv} = u\Delta t/\Delta x$
  internal wave speed c_{iw}: $\mathrm{CFL}^{c}_{iw} = c_{iw}\Delta t/\Delta x$
  external gravity wave $c_{ex} = \sqrt{gH}$: $\mathrm{CFL}^{c}_{ex} = c_{ex}\Delta t/\Delta x$
  diffusion: $\mathrm{CFL}^{diff} = \kappa\Delta t(2/\Delta x)^2$
  Coriolis: $\mathrm{CFL}^{cori} = f\Delta t$

- time-stepping criteria:
  Euler forward: CFL < 1
  AB-2: CFL $< 1/2$
  Euler Backward, Crank Nicolson: always stable
  DST advection: $\mathrm{CFL}^{adv} < 1$

- modified for multi-dimensional / multi-term problem. e.g.,
  3-D advection: $\mathrm{CFL}^{adv} = \Delta t \cdot \max(u/\Delta x, v/\Delta y, w/\Delta z)$
  3-D diffusion: $\mathrm{CFL}^{diff} = 4\,\Delta t\left(\kappa_x/\Delta x^2 + \kappa_y/\Delta y^2 + \kappa_z/\Delta z^2\right)$

No simple criteria for Non-Linear instability